Homework 5: Divide & Conquer

Due: October 17, 2024

Problem 1. The recurrence $T(n) = 7T(n/2) + n^2$ describes the running time of an algorithm A. A competing algorithm A' has a running time of $T'(n) = aT'(n/4) + n^2$. What is the largest integer value for a such that A' is asymptotically faster than A?

Solution. □

Problem 2. The Euclidean algorithm is a method for computing the greatest common divisor (gcd) of two numbers, taking advantage of the fact

$$\gcd(a,b) = \gcd(b, a-b)$$

for positive integers a, b satisfying $a \geq b$. Consider the following variation:

$$\gcd(a,b) = \begin{cases} 2\gcd(a/2, b/2) & \text{if } a, b \text{ even} \\ \gcd(a, b/2) & \text{if } a \text{ odd } b \text{ even} \\ \gcd((a-b)/2, b) & \text{if } a, b \text{ odd} \end{cases}$$

a. Prove the variation is correct.

b. Provide an algorithm that uses the variation to compute the greatest common divisor of two numbers a, b in $O(\log(ab))$

Solution. □

Problem 3. Given an n-bit binary integer, design a divide-and-conquer algorithm to convert it into its decimal representation. For simplicity, you may assume that n is a power of 2.

1. Provide a succinct (but clear) description of your algorithm, including pseudocode.
2. Prove the correctness of your algorithm.
3. Analyze the running time of your algorithm. Assume that it is possible to multiply two decimal integers numbers with at most m digits in $O(m^{\log_2 3})$ time.

Hint: An n-bit binary integer x can be expressed as $x = (x_{n-1}, x_{n-2}, \cdots, x_1, x_0)_2$ where $x_i \in \{0, 1\}$. Let $x_\ell = (x_{n/2-1}, x_{n/2-2}, \cdots, x_1, x_0)_2$ be the $(n/2)$-bit binary integer corresponding to the $(n/2)$ least significant digits of x. Let $x_m = (x_{n-1}, x_{n-2}, \cdots, x_{n/2+1}, x_{n/2})_2$ be the $(n/2)$-bit binary integer representing the $(n/2)$ most significant digits of x. Then, $x = x_\ell + 2^{n/2} \cdot x_m$. This should suggest us a way to set up a divide and conquer strategy… :) Careful about the number of subproblems!

Solution. □