Homework 10: Complexity Theory

Due: November 25, 2025

Problem 1. Compute the 3-CNF equivalent of the Boolean expression

$$x_1 \cap (x_3 \cup x_2 \cup \bar{x_1} \cup x_6) \cap (x_2 \cup \bar{x_3}) \cap (x_7 \cup x_3 \cup \bar{x_1}).$$

Prove that the two expressions are Boolean equivalent.

Solution. Let us rewrite the expression using $\wedge$ for AND, $\vee$ for OR, and $\neg$ for NOT:

$$F = x_1 \wedge (x_3 \vee x_2 \vee \neg x_1 \vee x_6) \wedge (x_2 \vee \neg x_3) \wedge (x_7 \vee x_3 \vee \neg x_1).$$

We want an equisatisfiable 3-CNF formula

We can frame existing clauses into 3-CNF format:

- The unit clause x_1 is equivalent to

$$(x_1 \vee x_1 \vee x_1).$$

- The two-literal clause $(x_2 \vee \neg x_3)$ is equivalent to

$$(x_2 \vee \neg x_3 \vee x_2).$$

- The three-literal clause $(x_7 \vee x_3 \vee \neg x_1)$ is already in 3-CNF form.

For the 4-lliteral clause: Consider the clause

$$C := (x_3 \vee x_2 \vee \neg x_1 \vee x_6).$$

Introduce an auxiliary variable y and replace C by the conjunction

$$C' := (x_3 \vee x_2 \vee y) \wedge (\neg y \vee \neg x_1 \vee x_6).$$

Each of these clauses has exactly 3 literals, so they are in 3-CNF.

We now define the 3-CNF formula:

$$\Phi = (x_1 \vee x_1 \vee x_1) \wedge (x_3 \vee x_2 \vee y) \wedge (\neg y \vee \neg x_1 \vee x_6) \wedge (x_2 \vee \neg x_3 \vee x_2) \wedge (x_7 \vee x_3 \vee \neg x_1).$$

Equisatisfiability: We claim that F and Φ are *equisatisfiable* (they agree on the existence of a satisfying assignment for the original variables $x_1, \ldots, x_7$).

- *(F satisfiable $\Rightarrow \Phi$ satisfiable)*
  Let α be a satisfying assignment for $x_1, \ldots, x_7$ such that $F(\alpha) = \text{T}$. In particular,

$$C(\alpha) = (x_3 \vee x_2 \vee \neg x_1 \vee x_6)(\alpha) = \text{T}.$$

There are two cases:

- If $(x_3 \vee x_2)(\alpha) = \text{T}$, set $y = \text{F}$. Then

$$(x_3 \vee x_2 \vee y)(\alpha, y) = \text{T}, \quad (\neg y \vee \neg x_1 \vee x_6)(\alpha, y) = (\text{T} \vee \dots) = \text{T}.$$

- If $(x_3 \vee x_2)(\alpha) = \text{F}$, then either $\neg x_1$ or x_6 is true under α. Set $y = \text{T}$. Then

$$(x_3 \vee x_2 \vee y)(\alpha, y) = \text{T}, \quad (\neg y \vee \neg x_1 \vee x_6)(\alpha, y) = (\text{F} \vee \neg x_1 \vee x_6)(\alpha) = \text{T}.$$

In both cases, we can extend α to α' on $(x_1, \dots, x_7, y)$ such that all clauses in Φ are satisfied. Thus Φ is satisfiable.

- (Φ *satisfiable* $\Rightarrow$ F *satisfiable*)
Let β be a satisfying assignment for $(x_1, \dots, x_7, y)$ with $\Phi(\beta) = \text{T}$. Then the two clauses replacing C,

$$(x_3 \vee x_2 \vee y), \quad (\neg y \vee \neg x_1 \vee x_6),$$

are both true under β. If $(x_3 \vee x_2)(\beta)$ is true, then clearly $C(\beta)$ is true. Otherwise, $(x_3 \vee x_2)(\beta)$ is false, so

$$(x_3 \vee x_2 \vee y)(\beta) = \text{T} \implies y(\beta) = \text{T},$$

and then

$$(\neg y \vee \neg x_1 \vee x_6)(\beta) = \text{T} \implies (\neg x_1 \vee x_6)(\beta) = \text{T}.$$

Hence at least one of $x_3, x_2, \neg x_1, x_6$ is true, so $C(\beta)$ is true. The other clauses of F are just the padded versions of those in Φ, so they also hold under the restriction of β to $x_1, \dots, x_7$. Thus F is satisfiable whenever Φ is.

Therefore F and Φ are equisatisfiable, and Φ is a 3-CNF form of F. $\square$

Problem 2. A vertex cover of a graph $G = (V, E)$ is a set of vertices $D \subseteq V$ that includes at least one endpoint of every edge of the graph.

The k-vertex-cover problem is the decision problem: does a graph $G(V, E)$ have a vertex cover of size k?

Prove the k-vertex-cover problem is NP-complete. [Hint: prove that G has a k-vertex-cover iff G has an independent set of size $|V| - k$.]

Solution. We consider the language

$$\textsf{VERTEX-COVER} = \{\langle G, k\rangle \mid G \text{ has a vertex cover of size } \leq k\}.$$

- k-VERTEX-COVER is in NP"
  A certificate for an instance $\langle G, k\rangle$ is a subset $D \subseteq V(G)$ with $|D| \leq k$. The verifier:

  1. checks that $|D| \leq k$;
  2. for each edge $(u, v) \in E(G)$, checks that $u \in D$ or $v \in D$.

  This takes $O(|V| + |E|)$ time, which is polynomial. Thus k-VERTEX-COVER $\in \textsf{NP}$.

- NP-hardness via reduction from INDEPENDENT-SET:

  Recall:

  $$\textsf{INDEPENDENT-SET} = \{\langle G, \ell\rangle \mid G \text{ has an independent set of size } \geq \ell\}$$

  is NP-complete.

  Using: For any graph $G = (V, E)$ and $D \subseteq V$,

  $$D \text{ is a vertex cover} \iff V \setminus D \text{ is an independent set.}$$

  Proof:

  – ($\Rightarrow$) Suppose D is a vertex cover. Then every edge has at least one endpoint in D. If there were an edge (u, v) with both endpoints in $V \setminus D$, that edge would have no endpoint in D, contradicting that D is a vertex cover. Hence there is no edge with both endpoints in $V \setminus D$, so $V \setminus D$ is independent.

  – ($\Leftarrow$) Suppose $I = V \setminus D$ is an independent set. Then no edge has both endpoints in I. So every edge must have at least one endpoint in D, i.e., D is a vertex cover.

  Taking sizes, we get:

  $$\exists \text{ vertex cover } D \text{ with } |D| \leq k \iff \exists \text{ independent set } I \text{ with } |I| \geq |V| - k.$$

  Reduction: Given an instance $\langle G, \ell\rangle$ of INDEPENDENT-SET, construct the instance

  $$\langle G, k\rangle \quad \text{where} \quad k = |V(G)| - \ell$$

  of VERTEX-COVER. This transformation is clearly polynomial-time.

By the lemma,

$$G \text{ has an independent set of size } \geq \ell \iff G \text{ has a vertex cover of size } \leq |V| - \ell = k.$$

Thus $\langle G, \ell \rangle \in \textsf{INDEPENDENT-SET}$ iff $\langle G, k \rangle \in \textsf{VERTEX-COVER}$, so

$$\textsf{INDEPENDENT-SET} \leq_p \textsf{VERTEX-COVER}.$$

Since k-VERTEX-COVER is in NP and NP-hard, it is NP-complete. □

Problem 3. Given a set of numbers $S = \{s_1, \ldots, s_n\}$, the PARTITION problem is to decide whether there is a set $T \subset S$ such that

$$\sum_{s \in T} = \sum_{s \in S \setminus T}.$$

Prove that the PARTITION problem is NP-complete.

[Hint: Reduce SUBSET-SUM(X, t) to PARTITION. Add one new number q to S such that there is a partition of $X \cup \{q\}$ iff there is a solution to SUBSET-SUM(X, t).]

Solution. We consider

$$\mathsf{PARTITION} = \{S \mid S \text{ can be split into two subsets of equal sum}\}.$$

- PARTITION is in NP:
  A certificate is a subset $T \subseteq S$. The verifier:
  1. computes $a = \sum_{s \in T}$ and $b = \sum_{s \in S \setminus T}$;
  2. checks that $a = b$.

  Both sums can be computed in time polynomial in $|S|$ and the input size, so $\mathsf{PARTITION} \in \mathsf{NP}$.

- NP-hardness via reduction from SUBSET-SUM (from hint)
  Recall:

  $$\mathsf{SUBSET\text{-}SUM} = \{\langle X, t\rangle \mid X \text{ is a multiset of numbers and there exists } Y \subseteq X \text{ with } \sum_{y \in Y} = t\}$$

  is NP-complete.

  Let $\langle X, t\rangle$ be an instance of SUBSET-SUM, where

  $$X = \{x_1, \ldots, x_n\}.$$

  Let

  $$s = \sum_{i=1}^{n} x_i$$

  and define one additional number

  $$q = s - 2t.$$

  Construct the PARTITION instance

  $$S' = X \cup \{q\}.$$

  This construction is clearly polynomial-time.

  Correctness of the reduction:

  We must show:

  $$\langle X, t\rangle \in \mathsf{SUBSET\text{-}SUM} \iff S' \in \mathsf{PARTITION}.$$

- ($\Rightarrow$) Suppose there exists $Y \subseteq X$ with

$$\sum_{y \in Y} = t.$$

Let $Z = X \setminus Y$, so $\sum_{z \in Z} = s - t$. Consider the subset

$$T = Y \cup \{q\} \subseteq S'.$$

Then

$$\sum_{u \in T} = \sum_{y \in Y} + q = t + (s - 2t) = s - t,$$

while

$$\sum_{u \in S' \setminus T} = \sum_{z \in Z} = s - t.$$

Thus S' can be partitioned into T and $S' \setminus T$ of equal sum, so $S' \in \textsf{PARTITION}$.

- ($\Leftarrow$) Suppose S' has a partition into two subsets of equal sum; that is, there exists $T \subseteq S'$ such that

$$\sum_{u \in T} = \sum_{u \in S' \setminus T}.$$

Let the total sum of S' be

$$\Sigma = \sum_{u \in S'} u = s + q = s + (s - 2t) = 2s - 2t.$$

Then each side of the partition has sum

$$\Sigma/2 = s - t.$$

Observe that q must lie in exactly one of the two parts. Without loss of generality assume $q \in T$. Then we can write

$$T = Y \cup \{q\}, \quad \text{with } Y \subseteq X.$$

Hence

$$\sum_{u \in T} = \sum_{y \in Y} + q = s - t.$$

Plugging in $q = s - 2t$ gives

$$\sum_{y \in Y} + (s - 2t) = s - t \quad \Rightarrow \quad \sum_{y \in Y} = t.$$

Therefore $Y \subseteq X$ is a subset whose elements sum to t, so $\langle X, t \rangle \in \textsf{SUBSET-SUM}$.

Since PARTITION is in NP and NP-hard, it is NP-complete. $\square$